

Pointwise strong approximation of almost periodic functions in S^1

Włodzimierz Łenski and Bogdan Szal

University of Zielona Góra

Faculty of Mathematics, Computer Science and Econometrics

65-516 Zielona Góra, ul. Szafrana 4a, Poland

W.Lenski@wmie.uz.zgora.pl, B.Szal @wmie.uz.zgora.pl

Abstract

We consider the class $GM(2\beta)$ in pointwise estimate of the deviations in strong mean of S^1 almost periodic functions from matrix means of partial sums of their Fourier series.

Key words: Almost periodic functions; Rate of strong approximation; Summability of Fourier series

2000 Mathematics Subject Classification: 42A24

1 Introduction

Let S^p ($1 \leq p \leq \infty$) be the class of all almost periodic functions in the sense of Stepanov with the norm

$$\|f\|_{S^p} := \begin{cases} \sup_u \left\{ \frac{1}{\pi} \int_u^{u+\pi} |f(t)|^p dt \right\}^{1/p} & \text{when } 1 \leq p < \infty \\ \sup_u |f(u)| & \text{when } p = \infty. \end{cases}$$

Suppose that the Fourier series of $f \in S^p$ has the form

$$Sf(x) = \sum_{\nu=-\infty}^{\infty} A_{\nu}(f) e^{i\lambda_{\nu}x}, \quad \text{where } A_{\nu}(f) = \lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L f(t) e^{-i\lambda_{\nu}t} dt,$$

with the partial sums

$$S_{\gamma_k} f(x) = \sum_{|\lambda_{\nu}| \leq \gamma_k} A_{\nu}(f) e^{i\lambda_{\nu}x}$$

and that $0 = \lambda_0 < \lambda_{\nu} < \lambda_{\nu+1}$ if $\nu \in \mathbb{N} = \{1, 2, 3, \dots\}$, $\lim_{\nu \rightarrow \infty} \lambda_{\nu} = \infty$, $\lambda_{-\nu} = -\lambda_{\nu}$, $|A_{\nu}| + |A_{-\nu}| > 0$. Let $\Omega_{\alpha, p}$, with some fixed positive α , be the set of functions

of class S^p bounded on $U = (-\infty, \infty)$ whose Fourier exponents satisfy the condition

$$\lambda_{\nu+1} - \lambda_\nu \geq \alpha \quad (\nu \in \mathbb{N}).$$

In case $f \in \Omega_{\alpha,p}$

$$S_{\lambda_k} f(x) = \int_0^\infty \{f(x+t) + f(x-t)\} \Psi_{\lambda_k, \lambda_k + \alpha}(t) dt,$$

where

$$\Psi_{\lambda, \eta}(t) = \frac{2 \sin \frac{(\eta - \lambda)t}{2} \sin \frac{(\eta + \lambda)t}{2}}{\pi (\eta - \lambda) t^2} \quad (0 < \lambda < \eta, \quad |t| > 0).$$

Let $A := (a_{n,k})$ be an infinite matrix of real nonnegative numbers such that

$$\sum_{k=0}^{\infty} a_{n,k} = 1, \text{ where } n = 0, 1, 2, \dots \quad (1)$$

Let us consider the strong mean

$$H_{n,A,\gamma}^q f(x) = \left\{ \sum_{k=0}^{\infty} a_{n,k} |S_{\gamma_k} f(x) - f(x)|^q \right\}^{1/q} \quad (q > 0). \quad (2)$$

As measures of approximation by the quantity (2), we use the best approximation of f by entire functions g_σ of exponential type σ bounded on the real axis, shortly $g_\sigma \in B_\sigma$ and the moduli of continuity of f defined by the formulas

$$E_\sigma(f)_{S^p} = \inf_{g_\sigma} \|f - g_\sigma\|_{S^p},$$

$$\omega f(\delta)_{S^p} = \sup_{|t| \leq \delta} \|f(\cdot + t) - f(\cdot)\|_{S^p},$$

and

$$w_x f(\delta)_p := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p dt \right\}^{1/p},$$

$$G_x f(\delta)_{s,p} := \left\{ \sum_{k=0}^{[\pi/(\alpha\delta)]} \left(\frac{1}{(k+1)\delta} \int_{k\delta}^{(k+1)\delta} |\varphi_x(t)|^p dt \right)^{s/p} \right\}^{1/s}, \quad s > 1,$$

where $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$, respectively.

Recently, L. Leindler [4] defined a new class of sequences named as sequences of rest bounded variation, briefly denoted by $RBVS$, i.e.

$$RBVS = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leq K(a) |a_m| \text{ for all } m \in \mathbb{N} \right\}, \quad (3)$$

where here and throughout the paper $K(a)$ always indicates a constant depending only on a .

Denote by MS the class of nonnegative and nonincreasing sequences. The class of general monotone coefficients, GM , will be defined as follows (see [11]):

$$GM = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leq K(a) |a_m| \text{ for all } m \in \mathbb{N} \right\}. \quad (4)$$

It is obvious that

$$MS \subset RBVS \subset GM.$$

In [5, 11, 12, 13] was defined the class of β -general monotone sequences as follows:

Definition 1 Let $\beta := (\beta_n)$ be a nonnegative sequence. The sequence of complex numbers $a := (a_n)$ is said to be β -general monotone, or $a \in GM(\beta)$, if the relation

$$\sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leq K(a) \beta_m \quad (5)$$

holds for all m .

In the paper [13] Tikhonov considered, among others, the following examples of the sequences β_n :

- (1) ${}_1\beta_n = |a_n|$,
- (2) ${}_2\beta_n = \sum_{k=[n/c]}^{[cn]} \frac{|a_k|}{k}$ for some $c > 1$.

It is clear that $GM({}_1\beta) = GM$ and (see [13, Remark 2.1])

$$GM({}_1\beta + {}_2\beta) \equiv GM({}_2\beta).$$

Moreover, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all n , where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (3)-(5) for the sequences $\alpha_n := (a_{n,k})_{k=0}^{\infty}$.

Now we can give the conditions to be used later on. We assume that for all n

$$\sum_{k=m}^{2m-1} |a_{n,k} - a_{n,k+1}| \leq K \sum_{k=[m/c]}^{[cm]} \frac{a_{n,k}}{k} \quad (6)$$

holds if $\alpha_n = (a_{n,k})_{k=0}^{\infty}$ belongs to $GM({}_2\beta)$, for $n = 1, 2, \dots$

We have shown in [7] the following theorem:

Theorem 2 If $f \in \Omega_{\alpha,p}$ ($p > 1$), $p \geq q$, $\alpha > 0$, $(a_{n,k})_{k=0}^{\infty} \in GM({}_2\beta)$ for all n , (1) and $\lim_{n \rightarrow \infty} a_{n,0} = 0$ hold, then

$$\left\| H_{n,A,\gamma}^q f \right\|_{S^p} \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \omega^q f \left(\frac{\pi}{k+1} \right)_{S^p} \right\}^{1/q},$$

for $n = 0, 1, 2, \dots$, where $\gamma = (\gamma_k)$ is a sequence with $\gamma_k = \frac{\alpha k}{2}$.

In this paper we consider the class $GM(2\beta)$ in pointwise estimate of the quantity $H_{n,A,\gamma}^q f$ for $f \in S^1$. Thus we present some analog of the following result of P. Pych-Taberska (see [10, Theorem 5]):

Theorem 3 *If $f \in \Omega_{\alpha,\infty}$ and $q \geq 2$, then*

$$\left\| H_{n,A,\gamma}^q f \right\|_{S^\infty} \ll \left\{ \frac{1}{n+1} \sum_{k=0}^n \left[\omega f \left(\frac{\pi}{k+1} \right)_{S^\infty} \right]^q \right\}^{1/q} + \frac{\|f\|_{S^\infty}}{(n+1)^{1/q}},$$

for $n = 0, 1, 2, \dots$, where $\gamma = (\gamma_k)$ is a sequence with $\gamma_k = \frac{\alpha k}{2}$, $a_{n,k} = \frac{1}{n+1}$ when $k \leq n$ and $a_{n,k} = 0$ otherwise.

We shall write $I_1 \ll I_2$ if there exists a positive constant K , sometimes depended on some parameters, such that $I_1 \leq KI_2$.

2 Statement of the results

Let us consider a function w_x of modulus of continuity type on the interval $[0, +\infty)$, i.e. a nondecreasing continuous function having the following properties: $w_x(0) = 0$, $w_x(\delta_1 + \delta_2) \leq w_x(\delta_1) + w_x(\delta_2)$ for any $\delta_1, \delta_2 \geq 0$ with x such that the set

$$\Omega_{\alpha,p,s}(w_x) = \left\{ f \in \Omega_{\alpha,p} : \left[\frac{1}{\delta} \int_0^\delta |\varphi_x(t) - \varphi_x(t \pm \gamma)|^p dt \right]^{1/p} \ll w_x(\gamma) \right. \\ \left. \text{and } G_x f(\delta)_{s,p} \ll w_x(\delta) \text{ , where } \gamma, \delta > 0 \right\}$$

is nonempty.

We start with proposition

Proposition 4 *If $f \in \Omega_{\alpha,1,2}(w_x)$, $\alpha > 0$ and $0 < q \leq 2$, then*

$$\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^q \right\}^{1/q} \ll w_x \left(\frac{\pi}{n+1} \right) + E_{\alpha n/2}(f)_{S^1},$$

for $n = 0, 1, 2, \dots$

Our main results are following

Theorem 5 *If $f \in \Omega_{\alpha,1,2}(w_x)$, $\alpha > 0$, $0 < q \leq 2$, $(a_{n,k})_{k=0}^\infty \in GM(2\beta)$ for all n , (1) and $\lim_{n \rightarrow \infty} a_{n,0} = 0$ hold, then*

$$H_{n,A,\gamma}^q f(x) \ll \left\{ \sum_{k=0}^\infty a_{n,k} \left[w_x \left(\frac{\pi}{k+1} \right) + E_{\frac{\alpha k}{2^{1+[c]}}}(f)_{S^1} \right]^q \right\}^{1/q}$$

for some $c > 1$ and $n = 0, 1, 2, \dots$, where $\gamma = (\gamma_k)$ is a sequence with $\gamma_k = \frac{\alpha k}{2}$.

Theorem 6 *If $f \in \Omega_{\alpha,1,2}(w_x)$, $\alpha > 0$, $0 < q \leq 2$, $(a_{n,k})_{k=0}^\infty \in MS$ for all n , (1) and $\lim_{n \rightarrow \infty} a_{n,0} = 0$ hold, then*

$$H_{n,A,\gamma}^q f(x) \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x\left(\frac{\pi}{k+1}\right) + E_{\frac{\alpha k}{2}}(f)_{S^1} \right]^q \right\}^{1/q}$$

for $n = 0, 1, 2, \dots$, where $\gamma = (\gamma_k)$ is a sequence with $\gamma_k = \frac{\alpha k}{2}$.

Remark 1 *Since, by the Jackson type theorem*

$$E_\sigma(f)_{S^p} \ll \omega f\left(\frac{1}{\sigma}\right)_{S^p}$$

and

$$\left\| \left[\frac{1}{\delta} \int_0^\delta |\varphi(t) - \varphi(t \pm \gamma)| dt \right] \right\|_{S^p} \leq \omega f(\gamma)_{S^p},$$

$$\|G.f(\delta)_{2,p}\|_{S^p} \leq \omega f(\delta)_{S^p},$$

the analysis of the proof of Proposition 4 shows that, the estimate from Theorem 5 implies the estimate from Theorem 2 with $p \geq 2$ and $0 < q \leq 2$. Thus, taking $a_{n,k} = \frac{1}{n+1}$ when $k \leq n$ and $a_{n,k} = 0$ otherwise, in the case $p \in [2, \infty]$ we obtain the better estimate than this one from Theorem 3 with $q = 2$ [10].

3 Proofs of the results

3.1 Proof of Proposition 4

In the proof we will use the following function $\Phi_x f(\delta, \nu) = \frac{1}{\delta} \int_\nu^{\nu+\delta} \varphi_x(u) du$, with $\delta = \delta_n = \frac{\pi}{n+1}$ and its estimate from [6, Lemma 1, p. 218]

$$|\Phi_x f(\zeta_1, \delta_2)| \leq w_x(\zeta_1) + w_x(\zeta_2) \quad (7)$$

for $f \in \Omega_{\alpha,1,2}(w_x)$ and any $\zeta_1, \zeta_2 > 0$.

We can also note that by monotonicity in $q \in (0, 2]$

$$\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^q \right\}^{1/q} \leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^2 \right\}^{1/2}.$$

Moreover, for $n = 0$ our estimate is evident, therefore we give the estimate of the quantity $H_{n,A,\gamma}^q f(x)$ with $q = 2$ and $n > 0$, only.

Denote by $S_k^* f$ the sums of the form

$$S_{\frac{\alpha k}{2}} f(x) = \sum_{|\lambda_\nu| \leq \frac{\alpha k}{2}} A_\nu(f) e^{i\lambda_\nu x}$$

such that the interval $\left(\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2}\right)$ does not contain any λ_ν . Applying Lemma 1.10.2 of [8] we easily verify that

$$S_k^* f(x) - f(x) = \int_0^\infty \varphi_x(t) \Psi_k(t) dt,$$

where $\Psi_k(t) = \Psi_{\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2}}(t)$, i.e.

$$\Psi_k(t) = \frac{4 \sin \frac{\alpha t}{4} \sin \frac{\alpha(2k+1)t}{4}}{\alpha \pi t^2}$$

(see also [2], p.41). Evidently, if the interval $\left(\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2}\right)$ contains a Fourier exponent λ_ν , then

$$S_{\frac{\alpha k}{2}} f(x) = S_{k+1}^* f(x) - (A_\nu(f) e^{i\lambda_\nu x} + A_{-\nu}(f) e^{-i\lambda_\nu x}).$$

Analyzing the proof of Proposition 1.2.2 from [1, p. 8] we can write

$$\begin{aligned} |A_{\pm\nu}(f)| &= |A_{\pm\nu}(f - g_{\alpha\mu/2})| \\ &= \left| \lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L (f(t) - g_{\alpha\mu/2}(t)) e^{-i\lambda_\nu t} dt \right| \\ &\leq \left| \lim_{L \rightarrow \infty} \sup_{T \geq L} \frac{1}{T} \int_0^T |(f(t) - g_{\alpha\mu/2}(t)) e^{-i\lambda_\nu t}| dt \right| \\ &\leq \left| \lim_{L \rightarrow \infty} \sup_{T \geq L} \frac{1}{T} \int_0^T |f(t) - g_{\alpha\mu/2}(t)| dt \right| \\ &\leq \left| \lim_{L \rightarrow \infty} \sup_{T \geq L} \sup_{U \in \mathbb{R}} \frac{1}{T} \int_U^{U+T} |f(t) - g_{\alpha\mu/2}(t)| dt \right| \\ &= \|f - g_{\alpha\mu/2}\|_W \leq \|f - g_{\alpha\mu/2}\|_{S^1} = E_{\alpha\mu/2}(f)_{S^1}, \end{aligned}$$

for some $g_{\alpha\mu/2} \in B_{\alpha\mu/2}$, with $\alpha k/2 < \alpha\mu/2 < \lambda_\nu$, where $\|\cdot\|_W$ is the Weyl norm.

Therefore, the deviation

$$\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^2 \right\}^{\frac{1}{2}}$$

can be estimated from above by

$$\begin{aligned} &\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \int_0^\infty \varphi_x(t) \Psi_{k+\kappa}(t) dt \right|^2 \right\}^{1/2} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} (E_{\alpha k/2}(f)_{S^1})^2 \right\}^{1/2} \\ &\leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \int_0^\infty \varphi_x(t) \Psi_{k+\kappa}(t) dt \right|^2 \right\}^{1/2} + E_{\alpha n/2}(f)_{S^1}, \end{aligned}$$

where κ equals 0 or 1. Applying the Minkowski inequality we obtain

$$\begin{aligned}
& \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \int_0^\infty \varphi_x(t) \Psi_{k+\kappa}(t) dt \right|^2 \right\}^{1/2} \\
&= \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \left(\int_0^{\pi/\alpha} + \int_{\pi/\alpha}^\infty \right) \varphi_x(t) \Psi_{k+\kappa}(t) dt \right|^2 \right\}^{1/2} \\
&\leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_1(k)|^2 \right\}^{1/2} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_2(k)|^2 \right\}^{1/2}.
\end{aligned}$$

So, for the first term we have

$$\begin{aligned}
& \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_1(k)|^2 \right\}^{1/2} \leq \left\{ \frac{2^\kappa e^2}{n+1} \sum_{k=n}^{2n} \left(1 - \frac{1}{n+1} \right)^{2n+\kappa} |I_1(k)|^2 \right\}^{1/2} \\
&\leq \left\{ \frac{2^\kappa e^2}{n+1} \sum_{k=n}^{2n} \left(1 - \frac{1}{n+1} \right)^{k+\kappa} |I_1(k)|^2 \right\}^{1/2} \\
&\leq \left\{ \frac{2^\kappa e^2}{n+1} \sum_{k=0}^\infty \left(1 - \frac{1}{n+1} \right)^{k+\kappa} |I_1(k)|^2 \right\}^{1/2} \\
&= \left\{ \frac{2^\kappa e^2}{n+1} \sum_{k=0}^\infty \left(1 - \frac{1}{n+1} \right)^{k+\kappa} \left| \int_0^{\pi/\alpha} \varphi_x(t) \Psi_{k+\kappa}(t) dt \right|^2 \right\}^{1/2} \\
&= \left\{ \frac{2^\kappa e^2}{n+1} \int_0^{\pi/\alpha} \int_0^{\pi/\alpha} \varphi_x(u) \overline{\varphi_x(v)} \sum_{k=0}^\infty \left(1 - \frac{1}{n+1} \right)^{k+\kappa} \Psi_{k+\kappa}(u) \Psi_{k+\kappa}(v) dudv \right\}^{1/2} \\
&\ll \left\{ \frac{2^\kappa e^2}{n+1} \int_0^{\pi/\alpha} \int_0^u \varphi_x(u) \overline{\varphi_x(v)} \sum_{k=0}^\infty \left(1 - \frac{1}{n+1} \right)^{k+\kappa} \Psi_{k+\kappa}(u) \Psi_{k+\kappa}(v) dudv \right\}^{1/2} \\
&= \left\{ \frac{2^\kappa e^2}{n+1} \left(\frac{4}{\alpha\pi} \right)^2 \int_0^{\pi/\alpha} \int_0^u \frac{\varphi_x(u) \overline{\varphi_x(v)} \sin \frac{\alpha u}{4} \sin \frac{\alpha v}{4}}{u^2 v^2} \right. \\
&\quad \left. \sum_{k=0}^\infty \left(1 - \frac{1}{n+1} \right)^{k+\kappa} \sin \frac{\alpha u(2(k+\kappa)+1)}{4} \sin \frac{\alpha v(2(k+\kappa)+1)}{4} dudv \right\}^{1/2} \\
&\leq \left\{ \frac{2^\kappa e^2}{n+1} \left(\frac{4}{\alpha\pi} \right)^2 \int_0^{\pi/\alpha} \int_0^u \frac{\varphi_x(u) \overline{\varphi_x(v)} \sin \frac{\alpha u}{4} \sin \frac{\alpha v}{4}}{u^2 v^2} \right. \\
&\quad \left. \sum_{k=0}^\infty \left(1 - \frac{1}{n+1} \right)^k \sin \frac{\alpha u(2k+1)}{4} \sin \frac{\alpha v(2k+1)}{4} dudv \right\}^{1/2}
\end{aligned}$$

Taking $y = \frac{\alpha u}{2}$, $z = \frac{\alpha v}{2}$ and $r = 1 - \frac{1}{n+1}$ in the relation (see [3] and [9])

$$\begin{aligned} & \sum_{k=0}^{\infty} r^k \sin \frac{y(2k+1)}{2} \sin \frac{z(2k+1)}{2} \\ &= \frac{\sin \frac{y}{2} \sin \frac{z}{2} (1-r) \left[(1+r)^2 + 2r(\cos y + \cos z) \right]}{\left[(1-r)^2 + 4r \sin^2 \frac{y+z}{2} \right] \left[(1-r)^2 + 4r \sin^2 \frac{y-z}{2} \right]} \end{aligned}$$

and using the inequality $\sin \frac{(y+z)}{2} \geq \frac{y+z}{\pi}$ ($y+z \leq \pi$), we obtain

$$\begin{aligned} & \left| \sum_{k=0}^{\infty} \left(1 - \frac{1}{n+1} \right)^k \sin \frac{\alpha u(2k+1)}{4} \sin \frac{\alpha v(2k+1)}{4} \right| \\ & \ll \frac{1}{n+1} \frac{uv}{\left[(1-r)^2 + (u+v)^2 \right] \left[(1-r)^2 + (u-v)^2 \right]}. \end{aligned}$$

Hence, taking $u-v=t$, by the Gabisoniya idea [3]

$$\begin{aligned} & \frac{1}{n+1} \sum_{k=n}^{2n} |I_1(k)|^2 \\ & \ll \frac{1}{(n+1)^2} \int_0^{\pi/\alpha} \int_0^u \frac{|\varphi_x(u) \varphi_x(v)| dudv}{\left[(n+1)^{-2} + (u+v)^2 \right] \left[(n+1)^{-2} + (u-v)^2 \right]} \\ & \leq \frac{1}{(n+1)^2} \int_0^{\pi/\alpha} \int_0^u \frac{|\varphi_x(u) \varphi_x(v)| dudv}{\left[(n+1)^{-2} + u^2 \right] \left[(n+1)^{-2} + (u-v)^2 \right]} \\ & = \frac{1}{(n+1)^2} \int_0^{\pi/\alpha} \int_0^u \frac{|\varphi_x(u) \varphi_x(u-t)| dudt}{\left[(n+1)^{-2} + u^2 \right] \left[(n+1)^{-2} + t^2 \right]} \\ & \leq \frac{1}{(n+1)^2} \sum_{i=0}^{[\pi(n+1)/\alpha]} \sum_{j=0}^i \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} \frac{|\varphi_x(u) \varphi_x(u-t)| dudt}{(n+1)^{-2} (1+i^2) \left[(n+1)^{-2} + t^2 \right]} \\ & \leq \sum_{i=0}^{[\pi(n+1)/\alpha]} \sum_{j=0}^i \frac{(n+1)^2}{(1+i^2)(1+j^2)} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| du \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} |\varphi_x(u-t)| dt \\ & \leq \sum_{i=0}^{[\pi(n+1)/\alpha]} \sum_{j=0}^i \frac{(n+1)^2}{(1+i^2)(1+j^2)} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| du \int_{\frac{i}{n+1} - \frac{j}{n+1}}^{\frac{i+1}{n+1} - \frac{j+1}{n+1}} |\varphi_x(v)| dv \end{aligned}$$

$$\begin{aligned}
& \ll \sum_{i=0}^{[\pi(n+1)/\alpha]} \sum_{j=0}^i \frac{(n+1)^2}{(1+i^2)(1+j^2)} \\
& \quad \left[\left(\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| du \right)^2 + \left(\int_{\frac{i}{n+1} - \frac{j}{n+1}}^{\frac{i+1}{n+1} - \frac{j}{n+1}} |\varphi_x(v)| dv \right)^2 \right] \\
& \ll \sum_{i=0}^{[\pi(n+1)/\alpha]} \left(\frac{n+1}{1+i} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| du \right)^2 \\
& \quad + \sum_{i=0}^{[\pi(n+1)/\alpha]} \sum_{j=0}^i \frac{1}{(1+j)^2} \left(\frac{n+1}{1+i} \int_{\frac{i}{n+1} - \frac{j}{n+1}}^{\frac{i+1}{n+1} - \frac{j}{n+1}} |\varphi_x(v)| dv \right)^2 \\
& \ll \sum_{i=0}^{[\pi(n+1)/\alpha]} \left(\frac{n+1}{1+i} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| du \right)^2 \\
& \quad + \sum_{j=0}^{[\pi(n+1)/\alpha]} \frac{1}{(1+j)^2} \sum_{i=j}^{[\pi(n+1)/\alpha]} \left(\frac{n+1}{1+i} \int_{\frac{i}{n+1} - \frac{j}{n+1}}^{\frac{i+1}{n+1} - \frac{j}{n+1}} |\varphi_x(v)| dv \right)^2 \\
& \ll \sum_{i=0}^{[\pi(n+1)/\alpha]} \left(\frac{n+1}{1+i} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| du \right)^2 + \sum_{\nu=0}^{[\pi(n+1)/\alpha]} \left(\frac{n+1}{1+\nu} \int_{\frac{\nu-1}{n+1}}^{\frac{\nu+1}{n+1}} |\varphi_x(v)| dv \right)^2 \\
& \ll \sum_{i=0}^{[\pi(n+1)/\alpha]} \left(\frac{n+1}{1+i} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| du \right)^2 = \left[G_x f \left(\frac{1}{n+1} \right)_2 \right]^2 \ll \left[w_x \left(\frac{\pi}{n+1} \right) \right]^2.
\end{aligned}$$

For the second term, using the Lenski method [6], we obtain

$$\begin{aligned}
& \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_2(k)|^2 \right\}^{1/2} \\
& \leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \sum_{\mu=1}^{\infty} \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} [\varphi_x(t) - \Phi_x f(\delta_k, t)] \Psi_{k+\kappa}(t) dt \right|^2 \right\}^{1/2} \\
& \quad + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \sum_{\mu=1}^{\infty} \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \Phi_x f(\delta_k, t) \Psi_{k+\kappa}(t) dt \right|^2 \right\}^{1/2} \\
& = \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{21}(k)|^2 \right\}^{1/2} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{22}(k)|^2 \right\}^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
|I_{21}(k)| &\leq \frac{4}{\alpha\pi} \sum_{\mu=1}^{\infty} \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} |\varphi_x(t) - \Phi_x f(\delta_k, t)| t^{-2} dt \\
&\leq \frac{4}{\alpha\pi} \sum_{\mu=1}^{\infty} \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \left[\frac{1}{\delta_k t^2} \int_0^{\delta_k} |\varphi_x(t) - \varphi_x(t+u)| du \right] dt \\
&= \frac{4}{\alpha\pi} \frac{1}{\delta_k} \int_0^{\delta_k} \sum_{\mu=1}^{\infty} \left\{ \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \frac{1}{t^2} |\varphi_x(t) - \varphi_x(t+u)| dt \right\} du \\
&= \frac{4}{\alpha\pi} \frac{1}{\delta_k} \int_0^{\delta_k} \sum_{\mu=1}^{\infty} \left\{ \left[\frac{1}{t^2} \int_0^t |\varphi_x(s) - \varphi_x(s+u)| ds \right]_{t=\mu\pi/\alpha}^{t=(\mu+1)\pi/\alpha} \right. \\
&\quad \left. + 2 \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \left[\frac{1}{t^3} \int_0^t |\varphi_x(s) - \varphi_x(s+u)| ds \right] dt \right\} du \\
&\ll \left| \frac{1}{\delta_k} \int_0^{\delta_k} \sum_{\mu=1}^{\infty} \left\{ \frac{1}{[(\mu+1)\pi/\alpha]^2} \int_0^{(\mu+1)\pi/\alpha} |\varphi_x(s) - \varphi_x(s+u)| ds \right. \right. \\
&\quad \left. \left. - \frac{1}{[\mu\pi/\alpha]^2} \int_0^{\mu\pi/\alpha} |\varphi_x(s) - \varphi_x(s+u)| ds \right\} du \right| \\
&\quad + \frac{1}{\delta_k} \int_0^{\delta_k} \sum_{\mu=1}^{\infty} \left\{ \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \left[\frac{1}{t^3} \int_0^t |\varphi_x(s) - \varphi_x(s+u)| ds \right] dt \right\} du.
\end{aligned}$$

Since $f \in \Omega_{\alpha,1,2}(w_x)$, for any x

$$\lim_{\zeta \rightarrow \infty} \frac{1}{\zeta^2} \int_0^{\zeta} |\varphi_x(s) - \varphi_x(s+u)| ds \leq \lim_{\zeta \rightarrow \infty} \frac{1}{\zeta} w_x(u) \leq \lim_{\zeta \rightarrow \infty} \frac{1}{\zeta} w_x(\delta_k) \leq \lim_{\zeta \rightarrow \infty} \frac{1}{\zeta} w_x(\pi) = 0,$$

and therefore

$$\begin{aligned}
|I_{21}(k)| &\leq \frac{1}{\delta_k} \int_0^{\delta_k} \frac{\alpha}{\pi} \left[\frac{\alpha}{\pi} \int_0^{\pi/\alpha} |\varphi_x(s) - \varphi_x(s+u)| ds \right] du \\
&\quad + \frac{1}{\delta_k} \int_0^{\delta_k} w_x(u) du \sum_{\mu=1}^{\infty} \left\{ \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \frac{1}{t^2} dt \right\} \\
&\ll \frac{1}{\delta_k} \int_0^{\delta_k} w_x(u) du + w_x(\delta_k) \sum_{\mu=1}^{\infty} \frac{\alpha}{\pi \mu^2} \\
&\ll w_x(\delta_k).
\end{aligned}$$

Next, we will estimate the term $|I_{22}(k)|$. So,

$$\begin{aligned}
I_{22}(k) &= \frac{2}{\alpha\pi} \sum_{\mu=1}^{\infty} \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \frac{\Phi_x f(\delta_k, t)}{t^2} \frac{d}{dt} \left(-\frac{\cos \frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}} + \frac{\cos \frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}} \right) dt \\
&= \frac{2}{\alpha\pi} \sum_{\mu=1}^{\infty} \left[\frac{\Phi_x f(\delta_k, t)}{t^2} \left(-\frac{\cos \frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}} + \frac{\cos \frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}} \right) \right]_{t=\mu\pi/\alpha}^{t=(\mu+1)\pi/\alpha} \\
&\quad + \frac{2}{\alpha\pi} \sum_{\mu=1}^{\infty} \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \frac{d}{dt} \left(\frac{\Phi_x f(\delta_k, t)}{t^2} \right) \left(\frac{\cos \frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}} - \frac{\cos \frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}} \right) dt \\
&= I_{221}(k) + I_{222}(k)
\end{aligned}$$

Since $f \in \Omega_{\alpha,1,2}(w_x)$, for any x (using (7))

$$\begin{aligned}
&\lim_{\zeta \rightarrow \infty} \left| \frac{\Phi_x f(\delta_k, \frac{\pi}{\alpha}\zeta)}{[\frac{\pi}{\alpha}\zeta]^2} \left(-\frac{\cos \left[\frac{\pi\zeta}{2}(k+\kappa) \right]}{\frac{\alpha(k+\kappa)}{2}} + \frac{\cos \left[\frac{\pi\zeta}{2}(k+\kappa+1) \right]}{\frac{\alpha(k+\kappa+1)}{2}} \right) \right| \\
&\ll \lim_{\zeta \rightarrow \infty} \frac{w_x(\delta_k) + w_x(\frac{\pi}{\alpha}\zeta)}{\zeta^2 k} \ll \lim_{\zeta \rightarrow \infty} \frac{w_x(\delta_k) + \zeta w_x(\frac{\pi}{\alpha})}{\zeta^2 k} \ll w_x(\pi) \lim_{\zeta \rightarrow \infty} \frac{1+\zeta}{\zeta^2} = 0,
\end{aligned}$$

and therefore

$$\begin{aligned}
I_{221}(k) &= \frac{2}{\alpha\pi} \sum_{\mu=1}^{\infty} \left[\frac{\Phi_x f(\delta_k, \frac{\pi}{\alpha}(\mu+1))}{[\frac{\pi}{\alpha}(\mu+1)]^2} \left(-\frac{\cos \left[\frac{\pi}{2}(\mu+1)(k+\kappa) \right]}{\frac{\alpha(k+\kappa)}{2}} \right. \right. \\
&\quad \left. \left. + \frac{\cos \left[\frac{\pi}{2}(\mu+1)(k+\kappa+1) \right]}{\frac{\alpha(k+\kappa+1)}{2}} \right) \right. \\
&\quad \left. - \frac{\Phi_x f(\delta_k, \frac{\pi}{\alpha}\mu)}{[\frac{\pi}{\alpha}\mu]^2} \left(-\frac{\cos \left[\frac{\pi}{2}\mu(k+\kappa) \right]}{\frac{\alpha(k+\kappa)}{2}} + \frac{\cos \left[\frac{\pi}{2}\mu(k+\kappa+1) \right]}{\frac{\alpha(k+\kappa+1)}{2}} \right) \right] \\
&= -\frac{2}{\alpha\pi} \frac{\Phi_x f(\delta_k, \pi/\alpha)}{[\pi/\alpha]^2} \left(-\frac{\cos \left[\frac{\pi}{2}\mu(k+\kappa) \right]}{\frac{\alpha(k+\kappa)}{2}} + \frac{\cos \left[\frac{\pi}{2}\mu(k+\kappa+1) \right]}{\frac{\alpha(k+\kappa+1)}{2}} \right) \\
&= -\frac{4}{\pi^3} \Phi_x f(\delta_k, \pi/\alpha) \left(\frac{\cos \left[\frac{\pi}{2}\mu(k+\kappa+1) \right]}{k+\kappa+1} - \frac{\cos \left[\frac{\pi}{2}\mu(k+\kappa) \right]}{k+\kappa} \right).
\end{aligned}$$

Using (7), we get

$$|I_{221}(k)| \ll \frac{1}{k+1} |\Phi_x f(\delta_k, \pi/\alpha)| \leq \frac{1}{(k+1)} (w_x(\delta_k) + w_x(\pi/\alpha)).$$

Similarly

$$\begin{aligned}
I_{222}(k) &= \frac{2}{\alpha\pi} \sum_{\mu=1}^{\infty} \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \left(\frac{\frac{d}{dt} \Phi_x f(\delta_k, t)}{t^2} - \frac{2\Phi_x f(\delta_k, t)}{t^3} \right) \\
&\quad \cdot \left(\frac{\cos \frac{\alpha t(k+\kappa)}{2}}{\frac{\alpha(k+\kappa)}{2}} - \frac{\cos \frac{\alpha t(k+\kappa+1)}{2}}{\frac{\alpha(k+\kappa+1)}{2}} \right) dt
\end{aligned}$$

and

$$\begin{aligned}
|I_{222}(k)| &\ll \frac{8}{\alpha^2(k+1)\pi} \sum_{\mu=1}^{\infty} \left[\int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \frac{|\varphi_x(t+\delta_k) - \varphi_x(t)|}{\delta_k t^2} dt \right. \\
&\quad \left. + 2 \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \frac{|\Phi_x f(\delta_k, t)|}{t^3} dt \right] \\
&\leq \frac{8}{\alpha^2(k+1)\pi\delta_k} \sum_{\mu=1}^{\infty} \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \frac{|\varphi_x(t+\delta_k) - \varphi_x(t)|}{t^2} dt \\
&\quad + \frac{16}{\alpha^2(k+1)\pi} \sum_{\mu=1}^{\infty} \int_{\mu\pi/\alpha}^{(\mu+1)\pi/\alpha} \frac{w_x(\delta_k) + w_x(t)}{t^3} dt \\
&\ll \frac{1}{(k+1)\delta_k} w_x(\delta_k) + \frac{1}{k+1} \sum_{\mu=1}^{\infty} \left[\left(w_x(\delta_k) + w_x\left(\frac{\pi(\mu+1)}{\alpha}\right) \right) \frac{\alpha^2}{\pi^2 \mu^3} \right] \\
&\ll w_x(\delta_k) + \frac{1}{k+1} \left[w_x(\delta_k) \sum_{\mu=1}^{\infty} \frac{1}{\mu^3} + \sum_{\mu=1}^{\infty} \frac{w_x\left(\frac{\pi(\mu+1)}{\alpha}\right)}{\mu^3} \right] \\
&\ll w_x(\delta_k) + \frac{1}{k+1} \left(w_x(\delta_k) + w_x\left(\frac{2\pi}{\alpha}\right) \sum_{\mu=1}^{\infty} \frac{\mu+1}{\mu^3} \right) \\
&\ll w_x(\delta_k) + \frac{1}{k+1} \left(w_x(\delta_k) + w_x\left(\frac{2\pi}{\alpha}\right) \right).
\end{aligned}$$

Summing up

$$|I_2(k)| \ll w_x(\delta_k) + \frac{1}{k+1} \left(w_x(\delta_k) + w_x\left(\frac{\pi}{\alpha}\right) + w_x\left(\frac{2\pi}{\alpha}\right) \right),$$

whence

$$\begin{aligned}
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_2(k)|^2 \right\}^{1/2} &\ll \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left(w_x\left(\frac{\pi}{k+1}\right) + \frac{1}{k+1} w_x\left(\frac{\pi}{\alpha}\right) \right)^2 \right\}^{1/2} \\
&\ll \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left(w_x\left(\frac{\pi}{k+1}\right) \right)^2 \right\}^{1/2} \leq w_x\left(\frac{\pi}{n+1}\right)
\end{aligned}$$

and thus the desired result follows. \square

3.2 Proof of Theorem 5

For some $c > 1$

$$H_{n,A,\gamma}^q f(x) = \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^q + \sum_{k=2^{[c]}}^{\infty} a_{n,k} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^q \right\}^{1/q}$$

$$\begin{aligned}
&\ll \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^q \right\}^{1/q} + \left\{ \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \left| S_{\frac{\alpha k}{2}} f(x) - f(x) \right|^q \right\}^{1/q} \\
&= I_1(x) + I_2(x).
\end{aligned}$$

Using Proposition 4 and denoting the left hand side of the inequality from its by F_n , i.e. $F_n = w_x \left(\frac{\pi}{n+1} \right) + E_{\alpha n/2}(f)_{S^1}$, we get

$$\begin{aligned}
I_1(x) &\leq \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} \frac{k/2+1}{k/2+1} \sum_{l=k/2}^k \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^q \right\}^{1/q} \\
&\leq \left\{ 2^{[c]} \sum_{k=0}^{2^{[c]}-1} a_{n,k} \frac{1}{k/2+1} \sum_{l=k/2}^k \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^q \right\}^{1/q} \\
&\ll \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} F_{k/2}^q \right\}^{1/q}.
\end{aligned}$$

By partial summation, our Proposition 4 gives

$$\begin{aligned}
I_2^q(x) &= \sum_{m=[c]}^{\infty} \left[\sum_{k=2^m}^{2^{m+1}-2} (a_{n,k} - a_{n,k+1}) \sum_{l=2^m}^k \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^q \right. \\
&\quad \left. + a_{n,2^{m+1}-1} \sum_{l=2^m}^{2^{m+1}-1} \left| S_{\frac{\alpha l}{2}} f(x) - f(x) \right|^q \right] \\
&\ll \sum_{m=[c]}^{\infty} \left[2^m \sum_{k=2^m}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| F_{\alpha 2^m/2}^q \right. \\
&\quad \left. + 2^m a_{n,2^{m+1}-1} F_{\alpha 2^m/2}^q \right] \\
&= \sum_{m=[c]}^{\infty} 2^m F_{\alpha 2^m/2}^q \left[\sum_{k=2^m}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| + a_{n,2^{m+1}-1} \right].
\end{aligned}$$

Since (6) holds, we have

$$\begin{aligned}
&a_{n,s+1} - a_{n,r} \\
&\leq |a_{n,r} - a_{n,s+1}| \leq \sum_{k=r}^s |a_{n,k} - a_{n,k+1}| \\
&\leq \sum_{k=2^m}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| \ll \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \quad (2 \leq 2^m \leq r \leq s \leq 2^{m+1}-2),
\end{aligned}$$

whence

$$a_{n,s+1} \ll a_{n,r} + \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \quad (2 \leq 2^m \leq r \leq s \leq 2^{m+1} - 2)$$

and

$$\begin{aligned} 2^m a_{n,2^{m+1}-1} &= \frac{2^m}{2^m - 1} \sum_{r=2^m}^{2^{m+1}-2} a_{n,2^{m+1}-1} \\ &\ll \sum_{r=2^m}^{2^{m+1}-2} \left(a_{n,r} + \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \right) \\ &\ll \sum_{r=2^m}^{2^{m+1}-1} a_{n,r} + 2^m \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k}. \end{aligned}$$

Thus

$$I_2^q(x) \ll \sum_{m=[c]}^{\infty} \left\{ 2^m F_{\alpha 2^m/2}^q \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} + F_{\alpha 2^m/2}^q \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \right\}.$$

Finally, by elementary calculations we get

$$\begin{aligned} I_2^q(x) &\ll \sum_{m=[c]}^{\infty} \left\{ 2^m F_{\alpha 2^m/2}^q \sum_{k=2^{m-[c]}}^{2^{m+[c]}} \frac{a_{n,k}}{k} + F_{\alpha 2^m/2}^q \sum_{k=2^m}^{2^{m+1}} a_{n,k} \right\} \\ &\ll \sum_{m=[c]}^{\infty} F_{\alpha 2^m/2}^q \sum_{k=2^{m-[c]}}^{2^{m+[c]}} a_{n,k} \\ &= \sum_{m=[c]}^{\infty} F_{\alpha 2^m/2}^q \sum_{k=2^{m-[c]}}^{2^m-1} a_{n,k} + \sum_{m=[c]}^{\infty} F_{\alpha 2^m/2}^q \sum_{k=2^m}^{2^{m+[c]}} a_{n,k} \\ &\ll \sum_{m=[c]}^{\infty} \sum_{k=2^{m-[c]}}^{2^m-1} a_{n,k} F_{\alpha k/2}^q + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+[c]}} a_{n,k} F_{\frac{\alpha k}{2^{1+[c]}}}^q \\ &= \sum_{m=[c]}^{\infty} \sum_{k=2^{m-[c]}}^{2^m-1} a_{n,k} F_{\alpha k/2}^q + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+[c]-1}} a_{n,k} F_{\frac{\alpha k}{2^{1+[c]}}}^q + \sum_{m=[c]}^{\infty} F_{\frac{\alpha 2^m}{2}}^q a_{n,2^{m+[c]}} \\ &= \sum_{m=[c]}^{\infty} \sum_{r=1}^{[c]} \sum_{k=2^{m-r}}^{2^{m-r+1}-1} a_{n,k} F_{\alpha k/2}^q + \sum_{m=[c]}^{\infty} \sum_{r=0}^{[c]-1} \sum_{k=2^{m+r}}^{2^{m+r+1}-1} a_{n,k} F_{\frac{\alpha k}{2^{1+[c]}}}^q \\ &\quad + \sum_{m=[c]}^{\infty} F_{\frac{\alpha 2^m}{2}}^q a_{n,2^{m+[c]}} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{r=1}^{[c]} \sum_{k=2^{[c]}-r}^{\infty} a_{n,k} F_{\alpha k/2}^q + \sum_{r=0}^{[c]-1} \sum_{k=2^{[c]}+r}^{\infty} a_{n,k} F_{\frac{\alpha k}{2^{1+[c]}}}^q + \sum_{k=2^{2[c]}}^{\infty} a_{n,k} F_{\frac{\alpha k}{2^{1+[c]}}}^q \\
&\ll \sum_{k=0}^{\infty} a_{n,k} F_{\frac{\alpha k}{2^{1+[c]}}}^q.
\end{aligned}$$

Thus we obtain the desired result. \square

3.3 Proof of Theorem 6

If $(a_{n,k})_{k=0}^{\infty} \in MS$ then $(a_{n,k})_{k=0}^{\infty} \in GM(2\beta)$ and using Theorem 5 we obtain

$$\begin{aligned}
H_{n,A,\gamma}^q f(x) &\leq \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x \left(\frac{\pi}{k+1} \right) \right]^q \right\}^{1/q} \\
&\quad + \left\{ \sum_{k=0}^{\infty} \sum_{m=k2^{[c]}}^{(k+1)2^{[c]}-1} a_{n,m} \left[E_{\frac{\alpha m}{2^{1+[c]}}} (f)_{S^p} \right]^q \right\}^{1/q} \\
&\leq \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x \left(\frac{\pi}{k+1} \right) \right]^q \right\}^{1/q} + \left\{ \sum_{k=0}^{\infty} \sum_{m=k2^{[c]}}^{(k+1)2^{[c]}-1} a_{n,m} \left[E_{\frac{\alpha k}{2}} (f)_{S^p} \right]^q \right\}^{1/q} \\
&\leq \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x \left(\frac{\pi}{k+1} \right) \right]^q \right\}^{1/q} + \left\{ \sum_{k=0}^{\infty} 2^{[c]} a_{n,k2^{[c]}} \left[E_{\frac{\alpha k}{2}} (f)_{S^p} \right]^q \right\}^{1/q} \\
&\ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[w_x \left(\frac{\pi}{k+1} \right) + E_{\frac{\alpha k}{2}} (f)_{S^p} \right]^q \right\}^{1/q}
\end{aligned}$$

This ends our proof. \square

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